Anisotropic Goal-Oriented Mesh Adaptation in Firedrake

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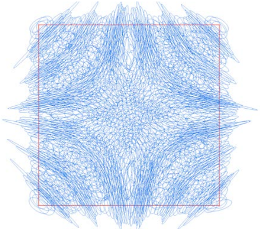
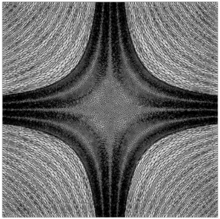
Firedrake '19, Durham

Metric-Based Mesh Adaptation.

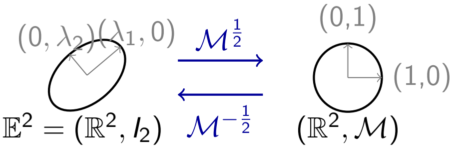
Riemannian **metric fields** $\mathcal{M} = \{\mathcal{M}(\mathbf{x})\}_{\mathbf{x}\in\Omega}$ are SPD $\forall \mathbf{x} \in \mathbb{R}^n$.
$\therefore$ Orthogonal eigendecomposition $\mathcal{M}(\mathbf{x}) = V \Lambda V^T$.

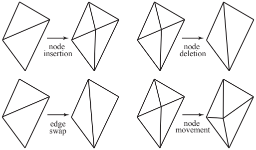


Steiner ellipses [Barral, 2015]



Resulting mesh [Barral, 2015]





Rokos and Gorman [2013]

The Hessian.

Consider interpolating $u \approx \mathcal{I}_h u \in \mathbb{P}1$.

It is shown in [Frey and Alauzet, 2005] that

$$\|u - \Pi_h u\|_{\mathcal{L}_\infty(K)} \leq \gamma \max_{\mathbf{x}\in K} \max_{\mathbf{e}\in\partial K} \mathbf{e}^T |H(\mathbf{x})| \mathbf{e}$$

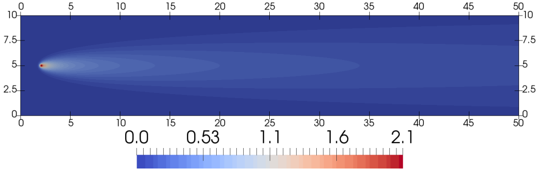
where $\gamma > 0$ is a constant related to the spatial dimension.

A metric tensor $\mathcal{M} = \{\mathcal{M}(\mathbf{x})\}_{\mathbf{x}\in\Omega}$ may be defined as

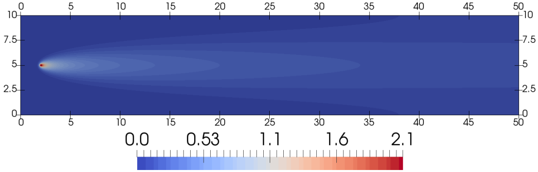
$$\mathcal{M}(\mathbf{x}) = \frac{\gamma}{\epsilon} |H(\mathbf{x})|,$$

where $\epsilon > 0$ is the tolerated error level.

Point Discharge with Diffusion



Analytical solution.



Finite element forward solution.

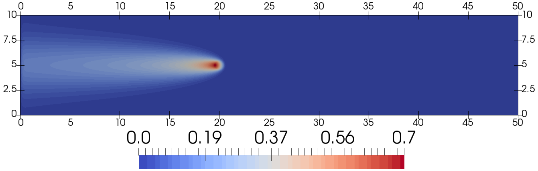
(Presented on a 1,024,000 element uniform mesh.)

Test case taken from **TELEMAC-2D Validation Document 7.0** [Riadh et al., 2014].

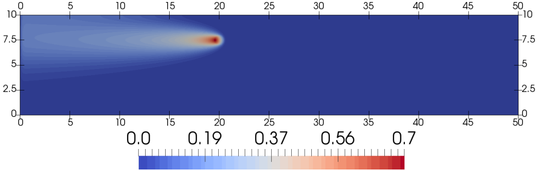
$$\begin{cases} \mathbf{u}\cdot\nabla\phi - \nabla\cdot(\nu\nabla\phi) & = f \\ \nu\widehat{\mathbf{n}}\cdot\nabla\phi|_{\text{walls}} & = 0 \\ \phi|_{\text{inflow}} & = 0 \end{cases}$$

$$f = \delta(x-2, y-5)$$

Point Discharge with Diffusion: Adjoint Problem



Adjoint solution for J_1.



Adjoint solution for J_2.

(Presented on a 1,024,000 element uniform mesh.)

$$J_i(\phi) = \int_{R_i} \phi \, \mathrm{d}x$$

$$R_1 = B_{\frac{1}{2}}((20, 5))$$

$$R_2 = B_{\frac{1}{2}}((20, 7.5))$$

Point Discharge with Diffusion: Convergence

Elements	$J_1(\phi)$	$J_1(\phi_h)$	$J_2(\phi)$	$J_2(\phi_h)$
4,000	0.20757	0.20547	0.08882	0.08901
16,000	0.16904	0.16873	0.07206	0.07205
64,000	0.16263	0.62590	0.06924	0.06922
256,000	0.16344	0.16343	0.06959	0.06958
1,024,000	0.16344	0.16345	0.06959	0.06958

$J_i(\phi)$: analytical solutions

$J_i(\phi_h)$: $\mathbb{P}1$ finite element solutions

Dual Weighted Residual (DWR).

Given a PDE $\Psi(u) = 0$ and its adjoint written in Galerkin forms

$$\rho(u_h, v) := L(v) - a(u_h, v) = 0, \quad \forall v \in V_h$$
$$\rho^*(u_h^*, v) := J(v) - a(v, u_h^*) = 0, \quad \forall v \in V_h$$

$\implies$ **a posteriori** error results [Becker and Rannacher, 2001]

$$J(u) - J(u_h) = \rho(u_h, u^* - u_h^*) + R^{(2)}$$
$$J(u) - J(u_h) = \frac{1}{2}\rho(u_h, u^* - u_h^*) + \frac{1}{2}\rho^*(u_h^*, u - u_h) + R^{(3)}$$

[Remainders $R^{(2)}$ and $R^{(3)}$ depend on errors $u - u_h$ and $u^* - u_h^*$.]

DWR Integration by Parts

$$J(u) - J(u_h) = \rho(u_h, u^* - u_h^*) + R^{(2)}$$

Applying integration by parts (again) *elementwise*:

$$|J(u) - J(u_h)|\Big|_K \approx |\langle \Psi(u_h), u^* - u_h^* \rangle_K + \langle \psi(u_h), u^* - u_h^* \rangle_{\partial K}|.$$

- $\Psi(u_h)$ is the strong residual on K;
- $\psi(u_h)$ embodies flux terms over elemental boundaries.

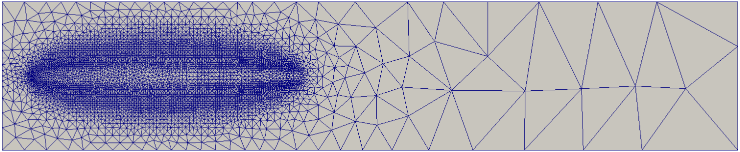
Isotropic Metric

$$|J(u)-J(u_h)|\Big|_K \approx \eta := |\langle\Psi(u_h), u^* - u_h^*\rangle_K + \langle\psi(u_h), u^* - u_h^*\rangle_{\partial K}|.$$

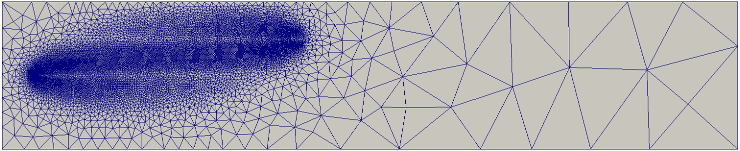
Isotropic case:

$$\mathcal{M} = \begin{bmatrix}\Pi_{\mathbb{P}1}\eta & 0\\ 0 & \Pi_{\mathbb{P}1}\eta\end{bmatrix}.$$

Isotropic Meshes



Centred receiver (12,246 elements).



Offset receiver (19,399 elements).

A posteriori Approach

Motivated by the approach of [Power et al., 2006], consider the interpolation error:

$$|J(u) - J(u_h)| \approx |\langle \Psi(u_h), \underbrace{u^* - u_h^*}_{u^* - \Pi_h u^*} \rangle_K + \langle \psi(u_h), \underbrace{u^* - u_h^*}_{u^* - \Pi_h u^*} \rangle_{\partial K}|$$

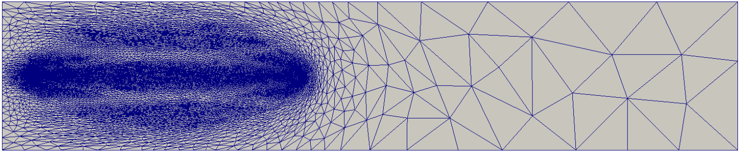
This suggests the node-wise metric,

$$\mathcal{M} = |\Psi(u_h)||H(u^*)|,$$

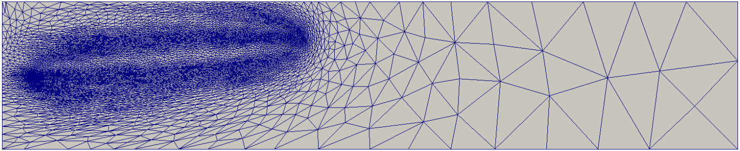
and correspondingly for the adjoint,

$$\mathcal{M} = |\Psi^*(u_h^*)||H(u)|.$$

A posteriori Anisotropic Meshes



Centred receiver (16,407 elements).



Offset receiver (9,868 elements).

A priori Approach

Alternative **a priori** error estimate [Loseille et al., 2010]:

$$J(u) - J(u_h) = \langle (\Psi_h - \Psi)(u), u^* \rangle + \widetilde{R}.$$

Assume we have the conservative form $\Psi(u) = \nabla \cdot \mathcal{F}(u)$, so

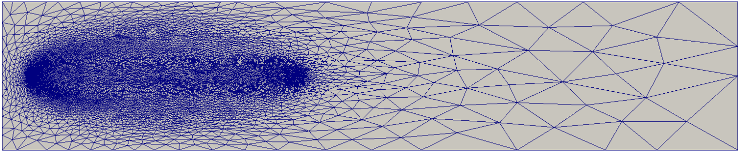
$$J(u) - J(u_h) \approx \langle (\mathcal{F} - \mathcal{F}_h)(u), \nabla u^* \rangle_\Omega - \langle \widehat{n} \cdot (\overline{\mathcal{F}} - \overline{\mathcal{F}}_h(u), u^*) \rangle_{\partial\Omega}.$$

This gives Riemannian metric fields

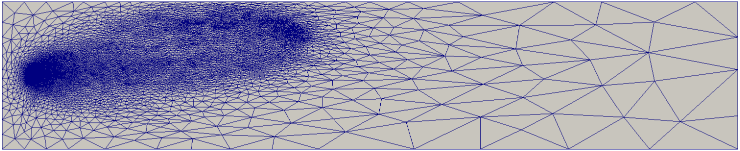
$$\mathcal{M}^{\text{volume}} = \sum_{i=1}^{n} |H(\mathcal{F}_i(u))| \left| \frac{\partial u^*}{\partial x_i} \right|,$$

$$\mathcal{M}^{\text{surface}} = |u^*| \left| \overline{H} \left(\sum_{i=1}^{n} \overline{\mathcal{F}}_i(u) \cdot n_i \right) \right|$$

A priori Anisotropic Meshes

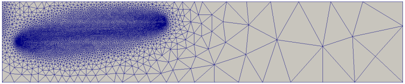


Centred receiver (44,894 elements).

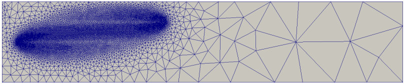


Offset receiver (29,143 elements).

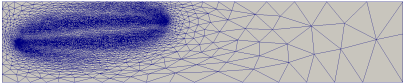
Meshes from Combined Metrics (Offset Receiver)



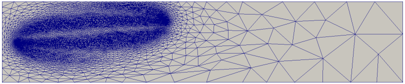
Averaged isotropic (13,980 elements).



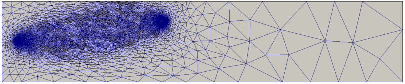
Superposed isotropic (19,588 elements).



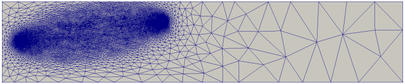
Averaged a posteriori (9,289 elements).



Superposed a posteriori (14,470 elems).

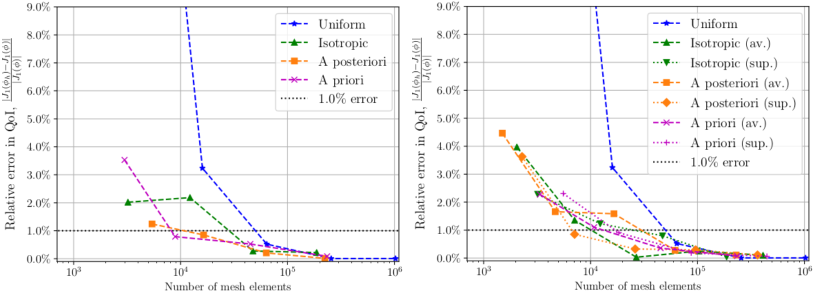


Averaged a priori (25,204 elements).

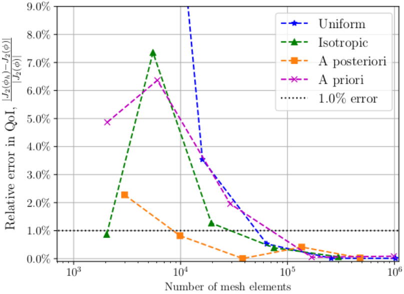


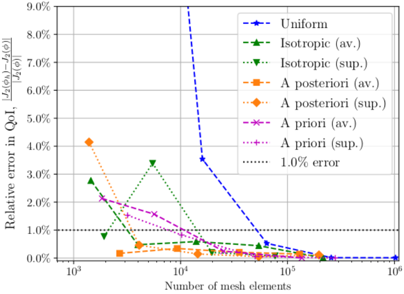
Superposed a priori (49,793 elements).

Convergence Analysis: Centred Receiver.

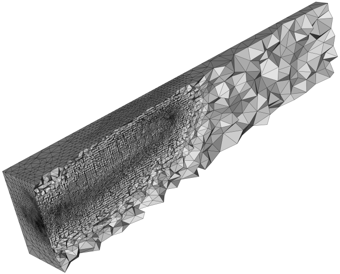


Convergence Analysis: Offset Receiver.

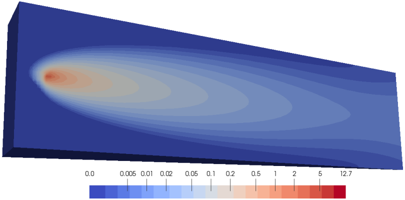




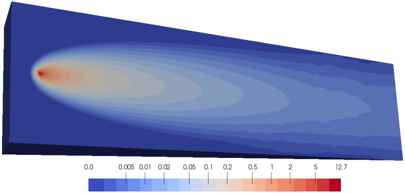
Three Dimensions.



Anisotropic mesh resulting from averaging a posteriori metrics.



Uniform mesh (1,920,000 elements).



Adapted mesh (1,766,396 elements).

Outlook.

To appear in proceedings of the 28^{th} International Meshing Roundtable:
JW, N Barral, D Ham, M Piggott, "Anisotropic Goal-Oriented Mesh Adaptation in Firedrake" (2019).

Future work:

- Time dependent (tidal) problems.
- Other finite element spaces, e.g. DG.
- Boundary and flux terms for anisotropic methods.
- Realistic desalination application.

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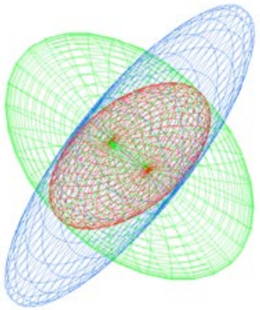
Metric Combination.

Consider metrics $\mathcal{M}_1$ and $\mathcal{M}_2$. How to combine these in a meaningful way?

Metric average:
$\mathcal{M} := \frac{1}{2}(\mathcal{M}_1 + \mathcal{M}_2)$.

Metric superposition:
intersection of Steiner ellipses.



Metric superposition [Barral, 2015]

Double $\mathcal{L}_2$ projection

We recover $H = \nabla^T \nabla u$ by solving the auxiliary problem

$$\begin{cases} H = \nabla^T \mathbf{g} \\ \mathbf{g} = \nabla u \end{cases}$$

as

$$\int_\Omega \tau : H_h \,\mathrm{d}x + \int_\Omega \mathrm{div}(\tau) : g_h \,\mathrm{d}x - \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\partial\Omega} (g_h)_i \tau_{ij} n_j \,\mathrm{d}s = 0, \quad \forall \tau$$

$$\int_\Omega \psi \cdot g_h \,\mathrm{d}x = \int_{\partial\Omega} u_h \psi \cdot \widehat{\mathbf{n}} \,\mathrm{d}s - \int_\Omega \mathrm{div}(\psi) u_h \,\mathrm{d}x, \quad \forall \psi.$$

Accounting for Source Term

Forward equation:

$$\nabla \cdot \mathcal{F}(\phi) = f, \quad \mathcal{F}(\phi) = \mathbf{u}\phi - \nu\nabla\phi.$$

$$\mathcal{M} = |H(\mathcal{F}_1(\phi))| \left|\frac{\partial \phi^*}{\partial x}\right| + |H(\mathcal{F}_2(\phi))| \left|\frac{\partial \phi^*}{\partial y}\right| + |H(f)|\,|\phi^*|.$$

Adjoint equation:

$$\nabla \cdot \mathcal{G}(\phi^*) = g, \quad \mathcal{G}(\phi^*) = -\mathbf{u}\phi^* - \nu\nabla\phi^*,$$

$$\mathcal{M} = |H(\mathcal{G}_1(\phi^*))| \left|\frac{\partial \phi}{\partial x}\right| + |H(\mathcal{G}_2(\phi^*))| \left|\frac{\partial \phi}{\partial y}\right| + |H(g)|\,|\phi|,$$